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HYDRODYNAMICS OF THE PROCESS OF FOAM FORMATION FROM A VISCOUS FLUID WITH BUBBLES*

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The process of expansion of a large number of gas bubbles in incompressible viscous liquid is considered. The free boundary shape is determined, and the formation of liquid films at convergence of two gas bubbles is investigated. A model of regularly packed bubbles is used for determining the process of their deformation and of formation of films.

Fundamentals of thermodynamics and static models of films were laid down in the works of Gibbs, Plato, and Rayleigh /1−3/. The hydrodynamics of films had attracted little attention.

1. Statement of the problem. Let an incompressible viscous liquid contain a large number of gas bubbles of the same volume V which grow with time owing to the expansion of gas or because of liberation of gas from the liquid. Function $V(t)$ is assumed known. The problem consists of determining the dependence of the shape of free boundaries on time. This implies the determination of formation of liquid films, their shape, and thickness.

Construction of an approximate definition of the problem can be attempted in the case of fairly low rates of bubbles volume variation, when the capillary

$$\frac{\mu}{\sigma}\frac{dR_v}{dt} \ll 1, \quad \frac{\rho R_v}{\sigma}\left(\frac{dR_v}{dt}\right)^2 \ll 1 \tag{1.1}$$

where R_v is the radius of a sphere of volume V. Then at the initial stage of bubble growth while the distance between their boundaries is small in comparison with their radii, their shape does not greatly differ form a sphere. Conditions (1.1) imply the smallness of dynamic stresses in comparison with capillary forces. At low Reynolds numbers $(dR/dt)R/\nu$ the first of conditions (1.1) is sufficient.

When the distance between /bubble/ surfaces is considerably smaller than the radii $h \ll R$, the thin layer approximation can be used. The dynamics of a film symmetric about the plane x_1, x_2 is defined by the equations /4/

$$\begin{aligned} &h\rho dv_i/dt = hF_i + h\nabla_i\,(^1/_2\sigma\Delta h + \Pi) + 2\nabla_j P_{ij}^{(s)} + \nabla_j\,[h\mu\,(2\delta_{ij}\,\mathrm{div}\,\mathbf{v} + \nabla_i v_j + \nabla_j v_i)]; \quad i, j = 1, 2 \\ &6\mu u_i = -h\nabla_j P_{ij}^{(s)}, \quad P_{ij}^{(s)} = \sigma\delta_{ij} + \lambda_s\,\mathrm{div}\,\mathbf{v}\delta_{ij} + \mu_s\,(\nabla_i v_j + \nabla_j v_i) \\ &\mathrm{div}\,(h\mathbf{u} + h\mathbf{v}) = -\partial h/\partial t \end{aligned} \tag{1.2}$$

where v is the surface velocity, u is the liquid velocity relative to the surface, averaged over the film cross section, σ is the surface tension coefficient, Π is the disjoining pressure (see, e.g., /5/), and μ_s, λ_s are the coefficients of the surface layer viscosity. Equations (1.2) are valid on conditions that the film thickness slowly varies along the coordinates $h \ll l$, where l is the characteristic scale of variation of h and flow parameters, the Reynolds number $h^2u/(\nu l) \ll 1$, and the characteristic time $\tau \gg h^2/\nu$. The force of gravity acts along the film, since the film length is fairly small $r \ll \sigma/(R\rho g)$ and distortions of its middle plane are immaterial.

The equations are closed by supplementary relations for the determination of surface tension σ which vary because of surface-active substances. We assume this effect to be of such importance that variations of an area element of the film surface are independent of its dynamics. To this corresponds a constant rate of surface expansion $\mathrm{div}\,\mathbf{v} = \mathrm{const}$ along the surface. The first of Eqs.(1.2) is then analogous to the Navier−Stokes equation, and the continuity equation is used for determining the thickness h. Unlike in the plane problem for the Navier−Stokes equation, here, for the determination of motion at the boundary it is necessary to specify two supplementary scale conditions, because the equations are of the fourth order in h. For example,

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four quantities v_1, v_2, h, Δh may be specified along some line Γ.

In the considered here problem the boundary conditions for Eqs.(1.2) are not a priori known, and must be determined using the conditions of merging the solution with the solution of equations in the region of large distances between bubble surfaces.

The free surface shape at fairly large distance from the adjacent bubble surface (the external region) is defined under conditions (1.1) by the equilibrium equation

$$\sigma\left(\frac{1}{R_1}+\frac{1}{R_2}\right)=p_\sigma=p_g-p_l^{(0)}+\rho(U-U_{(0)}) \tag{1.3}$$

where p_σ is the difference of pressures of gas p_g and liquid p_l, U is the potential of external mass forces, the subscript (0) corresponds to some point of the surface, and R_1 and R_2 are the principal curvature radii.

Equation (1.3) may be invalid in the region of small distances between buble surfaces, where it is necessary to consider (1.2). Let the film length between bubbles r considerably exceed its thickness and, also, the quantity $\sqrt{Rh}$. It is then possible to introduce the film boundary Γ, a plane contour along which the distance between bubble surfaces becomes zero relative to the large characteristic distance R in the external region.

If the bubble surface reaches the edge of the film at a fairly small angle α, the thin layer approximation is valid not only inside the contour Γ but, also, in some region outside it. At the boundary of region external to Γ the following conditions are to be specified for Eq.(1.3):

$$h|_\Gamma=0,\quad {}^1\!/_2\,\partial h/\partial n\,|_\Gamma=\operatorname{tg}\alpha \tag{1.4}$$

To determine motions of the film it is necessary to stipulate that at the limit $h/h_0\to\infty$ outside Γ the quantity h must approach the solution of the following equation with boundary condition on Γ:

$$ {}^1\!/_2\,\sigma\Delta h=p_\sigma,\quad h|_\Gamma=0 \tag{1.5}$$

which corresponds to specifying two boundary conditions for (1.2). Two further conditions are obtained from the condition of merging velocity v with values in the external region.

2. The jump of capillary pressure. In natural dimensionless notation in the boundary condition (1.5)

$$h=h_0h',\quad x=lx',\quad l=\sqrt{h_0\sigma/(2p_\sigma)} \tag{2.1}$$

where h_0 is the small thickness of film inside Γ, appears the small scale $l\sim\sqrt{Rh}$. The quantity $l\ll r$ because the film proper exists only under this condition /4,6/. Since the passage from the external region to the film proper (the inner region) takes place in a narrow region, and there is a jump there.

Analyzing the first of Eqs.(1.2) in variables of the small scale (2.1) and omitting small quantities of the order of Weber numbers h^2/l^2, l/r, l/R, we obtain

$$h\frac{\partial}{\partial x}\left(\frac{\sigma}{2}\frac{\partial^2h}{\partial x^2}+\Pi\right)+2\frac{\partial}{\partial x}P_{xx}^{(s)}=0,\quad P_{xx}^{(s)}=\sigma+(\lambda_s+2\mu_s)\frac{\partial v_n}{\partial x} \tag{2.2}$$

It can be shown that surface tension variation along the film is small $\Delta\sigma\ll\sigma$. Hence in the condition for the normal stress on the surface the coefficient $\sigma=\text{const}$.

Equation (2.2) admits the integral

$$2\sigma+\frac{\sigma h}{2}\frac{\partial^2h}{\partial x^2}-\frac{\sigma}{4}\left(\frac{\partial h}{\partial x}\right)^2+\int h\frac{\partial\Pi}{\partial h}\,dh+2(\lambda_s+2\mu_s)\frac{\partial v_n}{\partial x}=\text{const} \tag{2.3}$$

The velocity tangent to Γ is always continuous at the jump $v_\tau=\text{const}$, as can be seen from the theorem of momentum conservation in integral form.

For the equation of continuity in (1.2) at the jump, as well as in (2.2), the one-dimensional approximation is sufficient. Calculation of u yields

$$\frac{\partial h}{\partial t}-w_n\frac{\partial h}{\partial x}=-\frac{\partial}{\partial x}(hv_n)-\frac{\sigma}{24\mu}\frac{\partial}{\partial x}\left(h^3\frac{\partial^3h}{\partial x^3}\right)-\frac{\partial}{\partial x}\left(\frac{h^3}{12\mu}\frac{\partial\Pi}{\partial x}\right) \tag{2.4}$$

This equation is expressed in a system moving along the normal to Γ together with point $x=0$ of that contour, x is the distance along the normal to Γ, and w_n is the normal velocity of the point of contour Γ. To close system (2.2), (2.3) it is necessary either to have equations that link σ with the motion parameters and take v_n as the variable, or consider the case of incompressible surface. In the latter case the closing relation is $v_n = \text{const}$ along the coordinate.

The condition of merging the solution with the solution of the problem in the inner region is

$$h \to h_0, \quad x/l \to -\infty \tag{2.5}$$

On the basis of (1.5) the condition of passing to the external solution is

$$h \to h^{(e)}, \quad x/l \to \infty \tag{2.6}$$

and by virtue of (1.5) and definition of line Γ, the external solution is

$$h^{(e)} = \begin{cases} (p_\sigma/\sigma) x^2 + h_{\min}^{(e)}, & h_{\min}^{(e)} > 0 \\ (p_\sigma/\sigma) x^2 + 2\,\mathrm{tg}\,\alpha x, & h_{\min}^{(e)} < 0 \end{cases} \tag{2.7}$$

The line Γ is defined as the line of zero thickness $h^{(e)}$ or as the line of minimum of that thickness $(h_{\min}^{(e)} = O(h_0))$.

The variation of σ along the jump and its total change over it is, on the basis of (2.3), (2.5), and (2.6), without taking into account for brevity the surface viscosity, of the form

$$\sigma - \sigma_{(e)} + \frac{\sigma}{4} h \frac{\partial^2 h}{\partial x^2} - \frac{\sigma}{8}\left(\frac{\partial h}{\partial x}\right)^2 - \frac{1}{2}\int_h^\infty h \frac{\partial \Pi}{\partial h}\, dh = \Delta \tag{2.8}$$

$$\sigma_{(i)} - \sigma_{(e)} = \Delta + \frac{1}{2}\int_{h_0}^\infty h \frac{\partial \Pi}{\partial h}\, dh$$

$$\Delta = {}^1/_2 h_{\min}^{(e)} p_\sigma, \quad \Delta > 0; \quad \Delta = -{}^1/_2 \sigma\, \mathrm{tg}^2\,\alpha, \quad \Delta < 0$$

At the formation of the film, when contour Γ widens $(w_n > 0)$ or is stationary and thinning of the film takes place /6/, the film surface inside contour Γ is more stretched than outside it, so that in conformity with /1/ we have $\sigma_{(i)} > \sigma_{(e)}$. Then, as seen from (2.7) and (2.8), in the case of $\Pi \approx 0$ the external surface reaches the film along the tangent in scale R, and in the boundary condition (1.4) the angle $\alpha \equiv 0$.

An angle $\alpha \neq 0$ is only possible when $\Delta < 0$, and $\sigma_{(i)} < \sigma_{(e)}$. When $\Pi \approx 0$ (a film of macroscopic thickness) this is usually possible with compression of the surface inside contour Γ. However, even when angle α can be determined from the equation of the jump $(\sigma_{(i)} < \sigma_{(e)})$, its value remains negligibly small for the formulation of the equilibrium problem (1.3), if the thin layer approximation holds.

It is important that by virtue of (2.8) small variations of surface tension $\Delta \sim h_0 p_\sigma$ or $\Delta \sim \sigma\alpha^2$ always correspond to small film thicknesses h_0 and small angles α.

Quasi-steady approximation. Passing to dimensionless notation and time $t' = t/\tau$, where τ is the time scale of variation of contour Γ and capillary pressure p_σ, we obtain the condition of validity of the quasi-steady approximation $\tau \mid v_n - w_n \mid \gg l$. The characteristic time τ, scale s, and velocity w_n of contour Γ translation are interrelated $s \sim w_n \tau$, hence the last condition is equivalent to the inequality

$$l/s \ll (w_n - v_n)/w_n \tag{2.9}$$

which is satisfied when the velocity of contour Γ varies at large distance $s \gg l$ and when the relative velocity is not too low in comparison with w_n. At the formation of the film its dimension $r \gg l$ and condition (2.9) must be satisfied $(s \sim r)$.

Taking into account (2.5) and omitting the implicit dependence on time in (2.4), we obtain

$$w_n (h - h_0) = v_n h - v_{n0} h_0 + \frac{\sigma}{24\mu} h^3 \frac{\partial^3 h}{\partial x^3} + \frac{h^3}{12\mu} \frac{\partial \Pi}{\partial h} \frac{\partial h}{\partial x} \tag{2.10}$$

where v_{n0} is the value of v_n, generally a variable quantity, at the inner region boundary (as $x/l \to -\infty$).

Consider the limit case of the incompressible surface $v_n = \text{const}$, restricting the investigation to a macroscopically thick film, $\Pi = 0$ (similar problems were considered in /6/ for $\Pi \neq 0$. Substitution of variables (2.1) into (2.10), (2.5), and (2.7) so that

$$l = \sqrt{\frac{3h_0}{2p_\sigma}} = \frac{h_0}{\sqrt{c}}\left(\frac{\sigma}{24\mu\,(w_n - v_n)}\right)^{1/3} \qquad (2.11)$$

where c is so far an unknown quantity, results in the following problem for the determination of c:

$$y^3y''' - y + 1 = 0, \quad y\,|_{-\infty} = 1, \quad y''\,|_{+\infty} = c$$

A similar problem was considered in /3/ for a steady flow. Numerical calculation yields $c = 0.6429$, and from (2.11) we have

$$h_0 = 2.675\,\frac{\sigma}{p_\sigma}\left(\frac{\mu\,(w_n - v_n)}{\sigma}\right)^{2/3}, \quad w_n - v_n > 0 \qquad (2.12)$$

If the last inequality is satisfied, the film thickness is uniquely defined, in the opposite case the solution contains one arbitrary parameter, which means that h_0 is not a quantity to be determined but must be specified.

3. **Equations outside the jump.** Consider the regions inside and outside the contour Γ. Dimension of the inner region is considerably larger than the scale $r \gg l$. The scale of variation of functions with respect to the coordinate in the inner region is equal r, and with respect to time it is $\sim r/w$, and the characteristic thickness is $\sim h_0$. The comparison of (1.2) with this equation taken into account, with Eq.(2.2) at the jump shows with an accuracy to small $\sim l^2/r^2$ it is possible to disregard the contribution of Δh in (1.2). Omitting also small quantities of the order of the Weber number, we have

$$hF_i + h\nabla_i\Pi + 2\nabla_j P_{ij}^{(s)} = 0 \qquad (3.1)$$

When the surface viscous stresses and mass forces ($\rho grR \ll 2\sigma$) are small, (3.1) is integrable

$$2\sigma + \int h\,\frac{\partial\Pi}{\partial h}\,dh = \mathrm{const} \qquad (3.2)$$

In the case of macroscopic films the integral in (3.2) is to be neglected, and the surface tension in the inner region is constant. Variation of tension $\Delta\sigma_{(i)}$ is asymptotically small in comparison with the tension Δ at the jump, viz.

$$\Delta\sigma_{(i)} = O\left(\frac{l^2}{r^2}\Delta\right), \quad \Delta = (\sigma_{(i)} - \sigma_{(e)})\,|_\Gamma \qquad (3.3)$$

Note that Eq.(3.1) can be invalid at some sections of the inner region where $h \gg h_0$.

The equation of conservation of mass

$$\frac{\partial h}{\partial t} = -\frac{\sigma}{24\mu}\,\mathrm{div}\,(h^3\,\mathrm{grad}\,\Delta h) - \mathrm{div}\,(\mathbf{v}h) \qquad (3.4)$$

is simplified if there is the small parameter

$$\Omega = \frac{h_0^3}{s^4}\,\frac{\sigma\tau}{24\mu} \ll 1 \qquad (3.5)$$

where s is the scale of variation of parameters of motion of the film contour Γ, and τ is the respective time variation ($\tau \sim s/w_n$). In the region of scale s Eq.(3.4) in its principal approximation with respect to the small parameter Ω assumes the form

$$\partial h/\partial t = -\mathrm{div}\,(h\mathbf{v}) \qquad (3.6)$$

which determines the film thickness in the inner region where $s \sim r$.

Taking into account (2.11) makes it possible to represent condition (3.5) in the more convenient form

$$\Omega \sim \frac{l^3}{s^3}\,\frac{w_n - v_n}{w_n} \ll 1 \qquad (3.7)$$

Since $|w_n - v_n| \leqslant |w_n|$, hence condition (3.7) is satisfied, if condition (2.9) of the quasi-steadiness of jump (2.9) is satisfied.

In the region external to Γ, where the distance between free boundaries is large, the motion is defined by the Navier–Stokes equations in the region whose boundary is defined by the equilibrium equations(1.3). Taking into consideration that the scale of variation of functions is of the order of dimension R of the region, we shall estimate the maximum variation of surface tension σ under the action of the tangential stress

$$\Delta\sigma_{(e)} \sim \mu v \sim \sqrt{h/R}\,(\sigma_{(i)} - \sigma_{(e)})|_\Gamma \ll (\sigma_{(i)} - \sigma_{(e)})|_\Gamma \qquad (3.8)$$

It follows from (3.8) that in the case of thin films the tension in the external region is $\sigma_{(e)} = \mathrm{const}$. Thus the basic change of surface tension occurs in the jump region. By virtue of (3.3) the quantity $\sigma_{(i)} - \sigma_{(e)}$ along Γ is constant. This supplementary condition imposed on the solution at the jump indicates the constancy of basic parameters along the boundary if $w_n - v_n > 0$

$$(w_n - v_n)|_\Gamma = \mathrm{const}, \quad h_0|_\Gamma = \mathrm{const} \qquad (3.9)$$

In this case $h_0|\Gamma$ can be determined, if the quantity $w_n - v_n$ is known. When on Γ $w_n - v_n < 0$, the solution at the jump contains an additional parameter, and the quantity h_0 can vary along Γ, and can be taken as a given quantity.

The motion of liquid in the external region is induced by the change of the free surface shape which is defined by the quasi-steady equation (1.3) with variable parameters. It depends on the normal velocity of surface $v_n^{(e)}$ along the line of return of the free surface Γ. The tangential velocity along Γ must be bounded and is determined by solving the problem when the surface viscosity is zero. When the surface viscosity is non-zero, the tangential velocity along Γ must be specified.

4. Closing relations. It is generally necessary to consider the equations of transport of surface-active substances and take into account the isotherms of adsorbtion and surface tension. However, in important limit cases it is possible to do without these equations. If the effect of surface-active substances is considerable and the surface is incompressible under the action of viscous stresses, and the properties of the film surface do not differ from those of the surface of an infinitely deep liquid as regards variation of its area, the rate of change of surface elements is everywhere constant and corresponds to the respective rate of change of the bubble surface area S. In the film region we have

$$\mathrm{div}\,\mathbf{v} = S^{-1}dS/dt \qquad (4.1)$$

In the second case surface tension in the film region may to a considerably greater extent depend on the area element change, owing to its small thickness, than in the case of infinitely deep liquid surface. Then area elements of the film surface remain unchanged

$$\mathrm{div}\,\mathbf{v} = 0 \qquad (4.2)$$

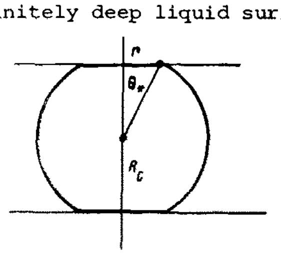

Fig.1

Formulas (1.3), (1.4) with $\alpha = 0$, and (2.10) or in the case of incompressible surface, (2.12), (3.6), and (3.9), as well as (4.1) or (4.2) constitute a closed definition in the case of one-dimensional problem. In multi-dimensional problems those formulas enable us to determine the total quantity of liquid in the film and its thickness along the edge depending on the time of its formation in the process.

5. The shape of surface in the external region. In a system with tightly packed bubbles a small radius of contour Γ of the contact area corresponds to a small deformation of a spherical bubble, and the film is of small size. This particularly occurs at the incipient film formation.

Let us consider on the assumption of deformation smallness the linearized problem of compression of a bubble of fixed volume V by forces $\mathbf{F}_\alpha$ ($\alpha = 1, 2, \ldots, 2N$) directed along the radius and representing the uniform additional pressure p_σ applied to small flat area elements of radius r

$$|\mathbf{F}_\alpha| = \pi r^2 p_\sigma; \quad \mathbf{F}_\alpha = -\mathbf{F}_{\alpha+N}, \quad \alpha = 1, \ldots, N$$

The normal displacement of bubble surface

$$U = \sum_{\alpha=1}^{N} U_\alpha \qquad (5.1)$$

where U_α is the displacement produced by the compression of the bubble between two planes by forces F_α, $F_{\alpha+N}$.

Below, we consider fairly small bubbles, when it is possible to neglect the effect of gravitation ($\rho g R^2 \ll 2\sigma$) on their shape.

The problem of deformation of a bubble between two planes is axially symmetric (Fig.1), hence in the domain $\theta \in (\theta_*, \pi - \theta_*)$ Eq.(1.3) in polar coordinates R, θ assumes the form

$$\frac{R^2 + 2R_\theta'^2 - RR_{\theta\theta}''}{(R^2 + R_\theta'^2)^{3/2}} + \frac{1 - \operatorname{ctg}\theta R_\theta'/R}{(R^2 + R_\theta'^2)^{1/2}} = \frac{p_\sigma}{\sigma} = \mathrm{const} \qquad (5.2)$$

Function $R(\theta)$ must be symmetric relative to $\pi/2$, $R(\theta - \pi/2) = R(-\theta + \pi/2)$, with a zero slope tangent when $\theta = \theta_*$. Consequently

$$\operatorname{ctg}\theta R_\theta' = R, \quad \theta = \theta_*, \qquad R(\theta_*)\cos\theta_* = R_c \qquad (5.3)$$

where R_c is equal half of the distance between planes.

The bubble volume is specified by

$$\frac{2\pi}{3}\int_{\theta_*}^{\pi-\theta_*} R^3 \sin\theta \, d\theta = V$$

For small deformations of bubbles $|R_\theta'/R| \ll 1$ and R is close to R_0

$$R = R_0 + \delta, \quad |\delta| \ll R_0, \quad R_0 = 2\sigma/p_\sigma$$

In the linear approximation

$$\delta_\theta'' + \delta_\theta' \operatorname{ctg}\theta + 2\delta = 0$$

equation (5.2) has the solution

$$\delta = a\left(\frac{\cos\theta}{2}\ln\frac{1+\cos\theta}{1-\cos\theta} - 1\right)$$

which is symmetric about $\pi/2$. Taking into account (5.3) we have

$$a = -R_c\theta_*^2, \quad R_0 = R_c(1 + \theta_*^2\ln(2/\theta_*) - {}^1/_2\,\theta_*^2)$$

In these formulas only the asymptotically principal terms are taken into account with $\theta_* \to 0$. The quantities R_0 and R_c are expressed in terms of the radius of a sphere of equivalent volume $R_v = (3V/(4\pi))^{1/3}$:

$$R_0 = (1 - {}^1/_3\,\theta_*^2)R_v, \quad R_c = (1 - \theta_*^2\ln(2/\theta_*) + {}^1/_6\,\theta_*^2)R_v \qquad (5.4)$$

The normal displacement of the contact area is equal

$$U_a = R_c - R_v = ({}^1/_6\,\theta_*^2 - \theta_*^2\ln(2/\theta_*))R_v \quad (\theta < \theta_*) \qquad (5.5)$$

The normal displacement of the surface away from the contact area is

$$U_a = -\left(\frac{\cos\theta}{2}\ln\frac{1+\cos\theta}{1-\cos\theta} - \frac{2}{3}\right)\theta_*^2 R_v \quad (\theta \gg \theta_*) \qquad (5.6)$$

The linearized equation differs from the exact one (5.2) by a small quantity of the order of $\theta_*^2\ln(1/\theta_*)$ when $\theta_* \to 0$.

6. Formation of films on regularly packed bubbles.

Let the bubble centers lie at the nodes of a regular grid, with the individual cells in the form of dodecahedrons. As the gas bubbles expand, the system of liquid and bubbles uniformly exapands. When the bubbles are symmetrically disposed, viscous forces act symmetrically at low Reynolds numbers, hence their resultant is zero and there is no relative displacement of bubbles.

In a rhombododecahedron (with all faces of the form of rhombuses) a normal to the face is at the angle of 60° to the normal lines to four pairs of opposite faces and at 90° to one of such lines. Hence from formulas (5.1), (5.5), and (5.6) for the displacement of an area element we have

$$U = R_c - R_v = -\left(\ln\frac{6R_v}{r} - \frac{7}{2}\right)\frac{r^2}{R_v} \qquad (6.1)$$

Considering that the cell volume $4\sqrt{2}R_c^3$ is equal $V + V_*$, where V_* is the volume of liquid, from (6.1) we have

$$\frac{dr}{dt} = \frac{R_v(1 - \pi/3\sqrt{2})}{2r(\ln(6R_v/r) - 4)}\frac{dR_v}{dt} \qquad (6.2)$$

For small r velocity v_n on Γ, determined by the bubble surface expansion, is of the small order r, hence

$$(w_n - v_n)\,|_\Gamma = dr/dt + O\,(rR_v^{\cdot}/R_v) \tag{6.3}$$

From (6.2), (6.3), and (2.12) for the dependence of the incipient film thickness on the distance from the bubble center we have the formula

$$h_0 = 0.685 R_v \left(\frac{\mu}{\sigma}\frac{dR_v}{dt}\right)^{2/3} \left(\frac{R_v/r}{\ln(6R_v/r) - 4}\right)^{2/3} \tag{6.4}$$

It follows from (6.4) that the film thickness substantially varies along the radius, with considerable thickening at the center. Its maximum thickness is determined by the condition $\sqrt{h_0 R_v} \sim r$ of appearance of the jump. This also yields the characteristic minimal dimension r at which the film is formed. By order of magnitude

$$\max h_0 \sim R_v\left(\frac{\mu}{\sigma}\frac{dR_v}{dt}\right)^{1/2}, \quad \min r \sim R_v\left(\frac{\mu}{\sigma}\frac{dR_v}{dt}\right)^{1/4}$$

Formula (6.4) implies that a fairly high convergence velocity of bubble surfaces at the instant of their collision is needed for obtaining thick films. The higher the rate of gas release, the thicker is the film. The same effect is produced by the liquid viscosity μ.

7. Formation of films in a developed foam. In the limit case of strongly deformed bubbles the liquid is concentrated in narrow filaments (the Plato–Gibbs channels) along the edges of polyhedrons. Everywhere, except at vertices, one curvature radius is smaller than the second $R_1 \ll R_2$. Taking into account (1.3) with $\rho g R_1^2 \ll \sigma$ and (1.4) with angle $\alpha = 0$ (Sect.2) we find that the cross section of channels is represented by three tangent circles of the same radius R_1, as is the case in the static problem /1,3/.

It is convenient to represent the volumes of liquid V_* and of bubble V by formulas in terms of the initial n_0 (when $V = 0$) or the current number n of bubbles in a unit of volume and, also, in terms of the volume part of liquid γ

$$n_0 V_* = 1, \; n\,(V + V_*) = 1, \; \gamma = V_*/(V + V_*), \; n = \gamma n_0 \tag{7.1}$$

Consider a foam consisting of regular polyhedrons. When $R_1 \ll R_v$ the half distance R_c between opposite films in a cell, and for the volume of liquid in the cell we have the approximate formulas

$$R_c = \left(\frac{\pi\sqrt{2}}{6}\right)^{1/3} R_v + O\left(\frac{R_1^2}{R_v}\right), \quad V_* = 6\sqrt{2}\left(1 - \frac{\pi}{2\sqrt{3}}\right) R_1^2 R_c$$

In the case when the film surface expansion takes place at the rate of gas bubble expansion (Sect.4) we have

$$(w_n - v_n)\,|_\Gamma = \frac{\sqrt{3}}{2}\frac{R_1}{R_v}\frac{dR_v}{dt}, \quad R_v = \left(\frac{3}{4\pi}V\right)^{1/3} \tag{7.2}$$

In the case of a nonexpandable film (Sect.4)

$$(w_n - v_n)\,|_\Gamma = \left(\frac{(\pi/3\sqrt{2})^{1/3}}{\sqrt{3}} + \frac{1}{2\sqrt{3}}\frac{R_1}{R_v}\right)\frac{dR_v}{dt} \tag{7.3}$$

Formulas (7.2) or (7.3) and (2.12) yield expressions for the formed film which differ from (6.4) by that the thickness variation is now linked only with the variation in time of R_1/R_v, $R_v^{\cdot}$. In the case of (7.2) the variation of film thickness is also due to its uniform expansion.

It is important that in the case of nonexpandable film (7.3) its thickness is much greater ($\sim (R_v/R_1)^{2/3}$ times) than in the case of the everywhere uniformly expanding bubble surface (7.2).

Formation of films ceases when velocity $R_v^{\cdot}$ drops to such an extent that condition (2.9) does no longer hold. After that it is necessary to solve the problem of thinning of the film.

Translated by J.J.D.

REFERENCES

1. GIBBS J.W., Collected Works, Vol.1. New York, Longmans Green, 1931.
2. RAYLEIGH J.W., Scientific Papers, Vol.3. Cambridge University Press, 1902.
3. MYSELS K.J., SHINODA K., and FRANKL S., Soap Films. London, in English, Pergamon Press, Book No. 09305, 1959.
4. VOINOV O.V., The equations of motion of free liquid films and the model of their plane-parallel thinning. Dokl. Akad. Nauk SSSR, Vol.216, No.2, 1974. (See also, in English, PMM Vol. 39, No.5, 1975.
5. ZONTAG G. and STRENGE K., Coagulation and Stability of Deisperse Systems /Russian translation/. Leningrad, KHIMIIA, 1973.
6. VOINOV O.V., On the theory of thinning of viscous fluid free films. PMTF, No.3, 1974.